MAT 4930: Quantum Algorithms and Complexity **Spring 2021**

Lecture 4: Qubits

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4.1 Basic definition

The smallest amount of quantum information is a qubit, which is a contraction of quantum bit. It generalizes the notion of the classical bit of information (i.e. a 0 or a 1).

Definition 4.1 (Qubit) *A qubit is a two-level quantum mechanical system.*

The above definition is vague because a qubit can take many shapes or forms. The description of the various physical systems that have been proposed for the realization of a qubit is outside the scope of this course. However, in the first lecture, we have hinted at the fact that the polarization of a photon could be used for a qubit (the two levels being $\uparrow$ and $\rightarrow$). Another example of a qubit is the spin of a spin $1/2$ particle. In this context the two levels are $\uparrow$ (spin up) and $\downarrow$ (spin down).

Once the qubit is defined, we need to establish how to model its *state*. The state of a classical bit can be either 0 or 1, but as we saw in the beam splitter experiement, the state of a quantum mechanical system is more complicated. In prticular, the photon coming out of the beam splitter is neither really in the $\uparrow$ state, nor in the $\rightarrow$ state. This means that there is a continuum of states in which a qubit can be. We model that in the following way.

Definition 4.2 (State of a qubit) *The state of a qubit is modeled by a vector $|\psi\rangle \in \mathbb{C}^2$ that satisfies $\||\psi\rangle\| = 1$.*

The reason why we impose the condition $\||\psi\rangle\| = 1$ becomes clear in Section 4.2 when we introduce the concept of measurement. Before that, let us look at a few examples of the state of a qubit.

Example 1 *Here are a few examples of state vectors of qubits. Note that this model is independant from the specific physical realization of the qubit in question.*

- $|\psi\rangle = |0\rangle$.
- $|\psi\rangle = |1\rangle$.
- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- $|\psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$.

4.2 Measurement of a qubit

The beam splitter experiement showed us that the probabilities of the photon taking a particular path did not add up via the regular probability laws a particule interpretation of light. Concretely, this means that the probability of observing the photon after the reconstruction of the beam is *not* given by the probabilities of observing the photon in each individual path. This is modeled in a counter-intuitive way: measuring the photon (in one of the two possible paths) actually alters its state (!!!). So the photon will not be the same after it is measured. As a popular quote attributed to Feynman says "If you think you understand quantum physics. then you probably don't". In this course, we ignore the phylosophical questions raised by this, and we focus on the mathematical modeling of measurement.

Measurement is done with respect to an observable. Observables are in correspondance with linear operators. The most famous one is the measurement of energy which corresponds to the Hamiltonian operator $\hat{H}$, and the state function ψ of a particule that satisfies the Schrdinger equation can be decomposed according to a basis of eigenfunctions of $\hat{H}$. Each of these eigenfunction can be labeled by the result of a measurement (in this case, an energy level or the particule).

The observables of a qubit can be modeled by orthonormal bases of $\mathbb{C}^2$, and each vector of such a basis identifies a possible outcome of the measurement. Without loss of generality, we usually assume that measurements are done if the *computational basis* $\{|0\rangle, |1\rangle\}$. While this is not going to be the case in this course, there are interesting questions about how measurements for different observables influence each other. When operators commute, they do not impact each other, but if they do not commute, the uncertainty principle limit the amount of information you can gain that way. In the following, we assume that measurements are done in the computational basis.

Definition 4.3 (Measurement postulate) *The measurement of $|\psi\rangle = a_0|0\rangle + a_i|1\rangle$ is the following process:*

$$|\psi\rangle \xrightarrow{\textit{Measurement}} |0\rangle \textit{ with probability } |a_0|^2, \quad \textit{or} \quad |1\rangle \textit{ with probability } |a_1|^2$$

and the information of which state the qubit collapses to.

We commonly say that when we measure in the basis $\{|0\rangle, |1\rangle\}$, we learn the classical information 0 or 1, and the state collapses to the corresponding basis vector. Here we also see why it was important to have $\| |\psi\rangle \| = 1$, otherwise the above model would not make sense.

Example 2 *Here are a few examples of measurements:*

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. *The measurement yields* 0 *with probability* 1/2 *and* 1 *with probability* 1/2.
- $|\psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$. *The measurement yields* 0 *with probability* $\cos^2(\theta)$ *and* 1 *with probability* $\sin^2(\theta)$.
- $|\psi\rangle = e^{i\alpha}(\cos(\theta)|0\rangle + \sin(\theta)|1\rangle)$. *The measurement yields* 0 *with probability* $\cos^2(\theta)$ *and* 1 *with probability* $\sin^2(\theta)$.

Note that in the last two examples, the states are different only by a factor $e^{i\alpha}$ which we call a *global phase.* Such a difference does not impact the measurement probabilities (although the quantum information that the state carries is still different and could impact the behavior of the qubit in the context of interference).

4.3 Bloch sphere representation

The representation of a qubit as a vector in $\mathbb{C}^2$ is not necessarily the most visually insightful. Indeed, most of us cannot visualize pairs of complex numbers. However, there is a way to map the state of a qubit to $\mathbb{R}^3$ where we have more intuition. As you know, $\mathbb{C}^2$ is not isomorphic to $\mathbb{R}^3$ (it is rather isomorphic to $\mathbb{R}^4$, but we loose a degree of freedom once we incorporate the following two constraints:

- Any state $a_0|0\rangle + a_1|1\rangle$ satisfies $|a_0|^2 + |a_1|^2 = 1$.
- Two states $|\psi_1\rangle$ and $|\psi_2\rangle$ that differ by a global phase are indistinguishable by way of measurement.

Definition 4.4 (Bloch sphere representation) *Let $|\psi\rangle$ be a qubit, we represent it on the unit sphere as given in Figure 4.3 by the polar coordinates $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$ such that*

$$|\psi\rangle \sim \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle,$$

where $|\psi_1\rangle \sim |\psi_2\rangle$ denotes the fact that the quantum states $|\psi_1\rangle, |\psi_2\rangle$ are equal modulo a global phase.

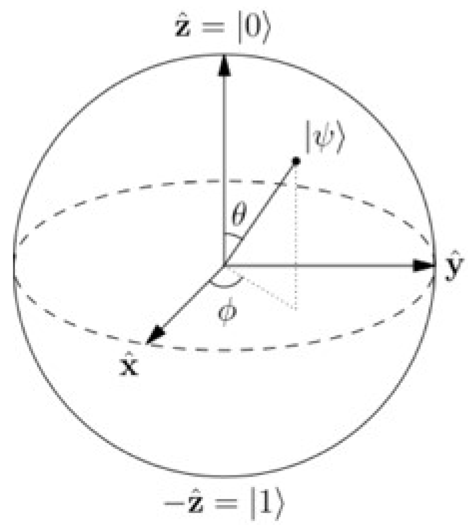

Figure 4.1: The Bloch sphere

Proposition 4.5 *If two states $|\psi_1\rangle, |\psi_2\rangle$ satisfy $|\psi_1\rangle \sim |\psi_2\rangle$, then their Bloch sphere representations are the same.*

Proof: Points on the unit sphere are uniquely defined by the two angles $0 \le \theta \le \pi$ and $0 \le \phi < 2\pi$. ■

How do we compute the angles θ, ϕ given $a_0, a_1 \in \mathbb{C}$ such that $|\psi\rangle = a_0|0\rangle + a_1|1\rangle$? Let $\rho_0, \rho_1 \in \mathbb{R}_{\ge 0}$, and $x_0, x_2 \in \mathbb{R}$ such that $a_0 = \rho_0 e^{ix_0}$ and $a_1 = \rho_1 e^{ix_1}$. Then we have

$$|\psi\rangle = \rho_0 e^{ix_0}|0\rangle + \rho_1 e^{ix_1}|1\rangle = e^{ix_0}\left(\rho_0|0\rangle + \rho_1 e^{i(x_1-x_0)}|1\rangle\right) \sim \rho_0|0\rangle + \rho_1 e^{i(x_1-x_0)}|1\rangle.$$

Now as $\rho_0, \rho_1 \ge 0$ and $\rho_0^2 + \rho_1^2 = 1$, there is a unique angle in $[0, \pi/2]$ such that they are respectively the cosine and sine of that angle. Let $\theta \in [0, \pi]$, such that $\rho_0 = \cos(\theta/2)$ and $\rho_1 = \sin(\theta/2)$, and let $\phi = x_1 - x_0 \bmod 2\pi$, then we have

$$|\psi\rangle \sim \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Example 3 *Let us compute the Bloch sphere representation* (θ, ϕ) *of the following states.*

- $|\psi\rangle = |0\rangle$. *We have* $(\theta, \phi) = (0, 0)$.
- $|\psi\rangle = |1\rangle$. *We have* $(\theta, \phi) = (\pi, 0)$.
- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. *We have* $(\theta, \phi) = (\pi/2, 0)$.
- $|\psi\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$ *for* $\alpha \in [\pi/2, \pi]$. *Here* $\cos(\alpha) \leq 0$ *and* $\sin(\alpha) \geq 0$. *Thus,* $-\cos(\alpha) = \cos(\pi - \alpha)$ *and* $\sin(\alpha) = \sin(\pi - \alpha)$, *and*

$$\begin{aligned} |\psi\rangle &= -1\left(-\cos(\alpha)|0\rangle - \sin(\alpha)|1\rangle\right) \\ &\sim -\cos(\alpha)|0\rangle - \sin(\alpha)|1\rangle \\ &= \cos(\pi - \alpha)|0\rangle - \sin(\pi - \alpha)|1\rangle \\ &= \cos(\pi - \alpha)|0\rangle + e^{i\pi}\sin(\pi - \alpha)|1\rangle. \end{aligned}$$

  Therefore, $\theta = 2(\pi - \alpha) \in [0, \pi]$ *and* $\phi = \pi$.